The countable existentially closed pseudocomplemented semilattice

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Abstract

As the class of pseudocomplemented semilattices is a universal Horn class generated by a single finite structure it has a $\aleph_0$-categorical model companion. We will construct the countable existentially closed pseudocomplemented semilattice which is the uniquely determined model of cardinality $\aleph_0$ of the model companion as a direct limit of algebraically closed pseudocomplemented semilattices.

1 Basic properties of pseudocomplemented semilattices and notation

A pseudocomplemented semilattice (PCSL) $\langle P; \wedge, ^*, 0\rangle$ is an algebra where $\langle P; \wedge\rangle$ is a meet-semilattice with least element 0, and for all $x, y \in P$, $x \wedge a = 0$ iff $x \leq a^*$. $1 := 0^*$ is obviously the greatest element of P. $x \parallel y$ is defined to hold if neither $x \leq y$ nor $y \leq x$ holds. An element d of P satisfying $d^* = 0$ is called *dense*, and if additionally $d \neq 1$ holds, then d is called a proper dense element. For $\mathbf{P} \in \mathcal{PCSL}$ the set $\mathrm{D}(\mathbf{P})$ denotes the subset of dense elements of $\mathbf{P}$, $\langle \mathrm{D}(\mathbf{P}); \wedge\rangle$ being a filter of $\langle P; \wedge\rangle$. An element s is called *skeletal* if $s^{**} = s$. The subset of skeletal elements of $\mathbf{P}$ is denoted by $\mathrm{Sk}(\mathbf{P})$. The abuse of notation $\mathrm{Sk}(x)$ for $x \in \mathrm{Sk}(\mathbf{P})$ should not cause ambiguities. Obviously, $\mathrm{Sk}(\mathbf{P}) = \{x^* \mid x \in P\}$.

For any pseudocomplemented semilattice $\mathbf{P}$ the pseudocomplemented semilattice $\widehat{\mathbf{P}}$ is obtained from $\mathbf{P}$ by adding a new top element. The maximal dense element of $\widehat{\mathbf{P}}$ different from 1 is denoted by e. Moreover, let $\mathbf{2}$ denote the two-element boolean algebra and $\mathbf{A}$ the countable atomfree boolean algebra. For a survey of pseudocomplemented semilattices consult [4] or [6].

If Σ is an axiomatization of a class of structures $\mathcal{K}$ with the model companion Σ^*, then a general result in model theory states that $\mathrm{Mod}(\Sigma^*) = \mathrm{Mod}(\Sigma)^{ec} = \mathcal{K}^{ec}$, $\mathcal{K}^{ec}$ denoting the class of the existentially closed members of $\mathcal{K}$, see e.g. [8]. As $\mathcal{PCSL}$ is a universal Horn class generated by a single finite structure it has an $\aleph_0$-categorical model companion Σ^*, see [2] for details. Constructing the unique countable model of Σ^* thus amounts to constructing a countable existentially closed PCSL.

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In [10] the following (semantic) characterization of algebraically closed p-semilattices is established:

Theorem 1.1. *A p-semilattice* $\mathbf{P}$ *is algebraically closed iff for any finite subalgebra* $\mathbf{F} \leq \mathbf{P}$ *there exists* $r, s \in \omega$ *and a PCSL* $\mathbf{F}'$ *isomorphic to* $\mathbf{2}^r \times \left(\widehat{\mathbf{A}}\right)^s$ *such that* $\mathbf{F} \leq \mathbf{F}' \leq \mathbf{P}$.

In [1] the following (syntactic) description of existentially closed p-semilattices is given:

Theorem 1.2. *A PCSL* $\mathbf{P}$ *is existentially closed iff* $\mathbf{P}$ *is algebraically closed and satisfies the following list of axioms:*

(EC1) iff

$$(\forall b_1, b_2)(\exists b_3)((Sk(b_1)\ \&\ Sk(b_2)\ \&\ b_1 < b_2) \to (Sk(b_3)\ \&\ b_1 < b_3 < b_2))$$

(EC2) iff

$$(\forall b_1, d)(\exists b_2)((Sk(b_1)\ \&\ D(d)\ \&\ b_1 < d\ \&\ b_1^* \parallel d) \to (Sk(b_2)\ \&\ b_1 < b_2 \parallel d\ \&\ b_2 < 1\ \&\ b_1 \dot\vee b_2^* < d\ \&\ b_1^* \wedge b_2 \parallel d))$$

(EC3) iff

$$(\exists d)(\mathsf{D}(d)\ \&\ d < 1)$$

(EC4) iff

$$(\forall d_1, d_2)(\exists d_3)((\mathsf{D}(d_1)\ \&\ d_1 < d_2) \to (d_1 < d_3 < d_2))$$

(EC5) iff

$$(\forall b, d_1)(\exists d_2)((D(d_1)\ \&\ Sk(b)\ \&\ 0 < b < d_1) \to (D(d_2)\ \&\ d_2 < d_1\ \&\ b \parallel d_2\ \&\ d_1 \wedge b^* = d_2 \wedge b^*))$$

For more background on PCLSs in general consult [4] and [6], for the notions concerning the problem tackled in this paper consult [1] and [9].

2 The countable existentially closed PCSL

As the objects of the direct limit we are going to construct we take $\{\mathbf{G}_n \mid n \in \mathbb{N} \setminus \{0\}\}$, where $\mathbf{G}_n := \left(\widehat{\mathbf{A}}\right)^n$. In view of Theorem 1.1 $\mathbf{G}_n$ is algebraically closed for all $n \in \mathbb{N} \setminus \{0\}$. We have to define embeddings $f_n : \mathbf{G}_n \to \mathbf{G}_{n+1}$ such that the direct limit of the directed family $\{\langle \mathbf{G}_m, g_{m,n}\rangle \mid m, n \in \mathbb{N}, 1 \leq m \leq n\}$ where $g_{i,j} := f_{j-1} \circ \cdots \circ f_i$ additionally satisfies (EC1)-(EC5) of Theorem 1.2.

In order to construct a direct limit with the desired properties we want the sequence $(f_n : \mathbf{G}_n \to \mathbf{G}_{n+1})_{n \in \mathbb{N}}$ to satisfy the following:

For all $n \geq 1$ the mapping $f_n : \mathbf{G}_n \to \mathbf{G}_{n+1}$ is an embedding such that the following holds:

1. For every anti-atom d of $D(\mathbf{G}_n)$ there is a $k \in \mathbb{N}$ such that $f_{n+k-1} \circ \cdots \circ f_n(d)$ is not an anti-atom of $\mathbf{G}_{n+k}$ anymore.
2. For $d = \min D(\mathbf{G}_n)$ there is an $l \in \mathbb{N}$ such that $f_{n+l-1} \circ \cdots \circ f_n(d) \neq \min D(\mathbf{G}_{n+l})$.

We define the embeddings $f_n : \mathbf{G}_n \to \mathbf{G}_{n+1}$, $n \geq 1$, as follows: Let $(\mathbf{G}_n)_i$ be the i-th factor of $\mathbf{G}_n$, $1 \leq i \leq n$, thus $(\mathbf{G}_n)_i = \widehat{\mathbf{A}}$. To determine $f_n(x)$ we distinguish between $x \in \mathrm{D}(\mathbf{G}_n)$ and $x \in \mathrm{Sk}(\mathbf{G}_n)$ as well as between n being even and n being odd.

Let $d_1, \ldots, d_n$ be an enumeration of the anti-atoms of $\mathrm{D}(\mathbf{G}_n)$, where $\mathrm{D}(\mathbf{G}_n) = \{e, 1\}^n$. For every anti-atom $d_i \in \mathrm{D}(\mathbf{G}_n)$ let $i_{\varphi_n(i)} \in \{1, \ldots, n\}$ be s.t. $(d_i)_k = e$ iff $k = \varphi_n(i)$, $1 \leq k \leq n$. That is $\varphi_n(i)$ is the place of the component of d_i that is e. Furthermore, let $a_{\varphi_n(i)1}, a_{\varphi_n(i)2}, \ldots$ be an enumeration of the elements of $(\mathrm{Sk}(\mathbf{G}_n))_{\varphi_n(i)} \setminus \{0, 1\}$ and let $U_{\varphi(i)j}$ be an ultrafilter on $(\mathrm{Sk}(\mathbf{G}_n))_{\varphi_n(i)}$ containing $a_{\varphi_n(i)1}$. To describe f_n we use the following notation: For $x = (x_1, \ldots, x_n) \in \mathbf{G}_n$ and $u \in \widehat{\mathbf{A}}$ we put $(\overrightarrow{x}, u) := (x_1, \ldots, x_n, u) \in \mathbf{G}_{n+1}$.

- If n is even put $f_n(d_1) := (\overrightarrow{d_1}, e)$, $f_n(d_i) := (\overrightarrow{d_i}, 1)$ for $i > 2$. For an arbitrary $d \in \mathrm{D}(\mathbf{G}_n)$ there is $I \subseteq \{1, \ldots, n\}$ such that $d = \bigwedge_{i \in I} d_i$. We put $f_n(d) := \bigwedge_{i \in I} f_n(d_i)$.

  $f_n(d_1)$ is not an anti-atom of $\mathbf{G}_{n+1}$ anymore whereas $f_n(d_i)$ still is an anti-atom of $\mathbf{G}_{n+1}$, $i = 2, \ldots, n$. These anti-atoms of $\mathbf{G}_{n+1}$ receive the numbers 1 to $n-1$, the two anti-atoms of $\mathrm{D}\left(\mathbf{G}_{n+1} \setminus f_n(\mathbf{G}_n)\right)$ receive n and $n+1$ according to the place of the e-component. This guarantees that for every anti-atom of $\mathbf{G}_n$ there is $k \in \mathbb{N}$ such that $f_{n+k-1} \circ \cdots \circ f_k(d)$ is not an anti-atom of $\mathbf{G}_{n+k}$ anymore. Thus 1. is satisfied.

  For $x = (x_1, \ldots, x_n) \in \mathrm{Sk}(\mathbf{G}_n)$ we put $f_n(x) := (\overrightarrow{x}, x_{\varphi_n(1)})$. Finally, for $x \in G_n$ arbitrary there is $d \in \mathrm{D}(\mathbf{G}_n)$ such that $x = x^{**} \wedge d$. We put $f_n(x) := f_n(x^{**}) \wedge f_n(d)$.

- Let n be odd. We define $f_n(x) := (\overrightarrow{x}, a)$ where

  $$a := \begin{cases} 1, & (x)_{\varphi_n(1)} \in U_{\varphi_n(1)1}; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

  where $a_{\varphi_n(1)1} = ((\mathbf{G})_n)_{\varphi_n(1)1}$ is denoted as the distinguished element for n.

  Then $f_n(d) = (\overrightarrow{d}, 1)$ for all $d \in D(\mathbf{G}_n)$: $(d)_{\varphi_n(1)} \in \{e, 1\} \subset U_{\varphi_n(1)1}$. Thus $f_n(d_i)$, $i = 1, \ldots, n$ are still anti-atoms of $\mathbf{G}_{n+1}$. They are numbered 1 to n, the anti-atom $(1, \ldots, 1, e)$ is numbered $n+1$. We obtain $f_n(\min \mathrm{D}(\mathbf{G}_n)) \neq \min D(\mathbf{G}_{n+1})$. Thus 2. is satisfied.

  The enumeration of $(\mathrm{Sk}(\mathbf{G})_{n+1})_i \setminus \{0, 1\}$, $1 \leq i \leq n+1$, is as follows: For $2 \leq i \leq n$ the enumeration is the same as for $(\mathrm{Sk}(\mathbf{G})_n)_i \setminus \{0, 1\}$. $(\mathrm{Sk}(\mathbf{G})_{n+1})_{n+1} \setminus \{0, 1\}$ can be enumerated arbitrarily. Let now x be an element of $(\mathrm{Sk}(\mathbf{G})_{n+1})_1 \setminus \{0, 1\} = (\mathrm{Sk}(\mathbf{G})_n)_1 \setminus \{0, 1\}$. Therefore, $x = a_{1j}$ is the jth element of $(\mathrm{Sk}(\mathbf{G})_n)_1 \setminus \{0, 1\}$ for a $j \in \mathbb{N}$. We distinguish three cases depending on the value of j. If $j = 1$ then x receives the number 2^n. If $2 \leq j \leq 2^n$, x receives the number $j - 1$. Finally, if $2^n < j$, then x receives the number j. This guarantees that every $x \in (\mathbf{G}_n)_i$, $1 \leq i \leq n$, becomes the distinguished element for some $n' \geq n$.

Setting $g_{n,n} = id_{\mathbf{G}_n}$ and $g_{i,j} = f_{j-1} \circ \cdots \circ f_i$ for $i < j$ we obtain the directed family $\{\langle \mathbf{G}_m, g_{m,n} \rangle \mid m, n \in \mathbb{N}, 1 \leq m \leq n\}$ of PCSL.

Claim. The direct limit $\mathbf{G}$ of the directed family $\{\langle \mathbf{G}_m, g_{m,n} \rangle \mid m, n \in \mathbb{N}, 1 \leq m \leq n\}$ of PCSLs is countable and existentially closed.

Proof. $\mathbf{G}$ is countable since a countable union of countable sets is countable. That $\mathbf{G}$ is algebraically closed follows from Theorem 1.1: Let $\mathbf{S}$ be a finite subalgebra of $\mathbf{G}$. According to the construction of $\mathbf{G}$ there is an $n \in \mathbb{N}$ such that there is a finite subalgebra $\mathbf{S}_1$ of $\mathbf{G}$ isomorphic to $\mathbf{G}_n = \left(\widehat{\mathbf{A}}\right)^n$ containing $\mathbf{S}$.

According to Theorem 1.2 it remains to show that $\mathbf{G}$ satisfies (EC1) - (EC5). (EC1) is satisfied as it is satisfied in $\mathbf{A}$. (EC3) is obviously satisfied. To prove the remaining three axioms we denote for $x \in \bigcup_{n=1}^{\infty} \mathbf{G}_n$ with $[x] \in \mathbf{G}$ the equivalence class of x.

For (EC4) consider arbitrary $d_1, d_2 \in \mathrm{D}(\mathbf{G})$ such that $d_1 < d_2$. There is $n \in \mathbb{N}$ and $x, y \in G_n$ s.t. $d_1 = [x]$, $d_2 = [y]$. There are $l \in \mathbb{N}$ and $z \in \mathrm{D}(\mathbf{G}_{n+l})$ such that $g_{n,n+l}(x) < z < g_{n,n+l}(y)$: We have $x = \bigwedge_{j \in J_x} x_j$, $y = \bigwedge_{j \in J_y} x_j$ for subsets $J_y \subsetneq J_x \subseteq \{1, \ldots, n\}$, x_j being an anti-atom of $\mathrm{D}(\mathbf{G}_n)$ for $j \in J_x$. For $j_0 \in J_x \setminus J_y$ there is according to Property 1 of the embeddings $f_m : \mathbf{G}_m \to \mathbf{G}_{m+1}$, $m \geq 1$, an $l \in \mathbb{N}$ such that $g_{n,n+l}(x_{j_0})$ is not an anti-atom of $\mathbf{G}_{n+l}$ anymore. Thus there is $u \in G_{n+l}$ with $g_{n,n+l}(x_{j_0}) < u < 1$ yielding $g_{n,n+l}(x) < g_{n,n+l}(y) \wedge u < g_{n,n+l}(y)$. Thus $d_1 = [x] = [g_{n,n+l}(x)] < [g_{n,n+l}(y) \wedge u] < [g_{n,n+l}(y)] = [y] = d_2$.

For (EC2) consider arbitrary $b_1 \in \mathrm{Sk}(\mathbf{G})$ and $d \in \mathrm{D}(\mathbf{G})$ s.t. $b_1 < d$ and $b_1^* \parallel d$. There is $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in G_n$ s.t. $b_1 = [x]$, $d = [y]$, $\mathrm{Sk}(x)$, $\mathrm{D}(y)$, $x < y$ and $x^* \parallel y$. We first assume that y is not an antiatom of $\mathbf{G}_n$. Then without loss of generality we can assume $x_1 = 0$, $y_1 = y_2 = e$. Then put $z := (1, x_2, 1, \ldots, 1)$ to obtain $x < z$, $z \parallel y$, $x^* \wedge z \parallel y$ and $x \dot{\vee} z^* < y$. Putting $b_2 := [z]$ yields what is requested in (EC2).

If y is an antiatom there is again according to Property 1 an $l \in \mathbb{N}$ such that $g_{n,n+l}(y)$ is not an antiatom of $\mathbf{G}_{n+l}$ anymore. For $x' := g_{n,n+l}(x)$ and $y' := g_{n,n+l}(y)$ we find as above $z \in G_{n+l}$ s.t. $x' < z$, $z \parallel y'$, $x^* \wedge z \parallel y'$ and $x' \dot{\vee} z^* < y'$. Putting $b_2 := [z]$ yields what is requested in (EC2) because $[x] = [x']$, $[y] = [y']$.

For (EC5) consider arbitrary $b \in \mathrm{Sk}(\mathbf{G})$ and $d_1 \in \mathrm{D}(\mathbf{G})$ s.t. $0 < b < d_1$. There is $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in G_n$ s.t. $b = [x]$, $d_1 = [y]$, $\mathrm{Sk}(x)$, $\mathrm{D}(y)$, $0 < x < y$. Let us assume that there is no $z \in \mathrm{D}(\mathbf{G}_n)$ s.t. $z < y$, $x || z$ and $x^* \wedge y = x^* \wedge z$, since otherwise we put $d_2 := [z]$.

According to the definition of the embeddings $\{f_m : \mathbf{G}_m \to \mathbf{G}_{m+1} \mid m \in \mathbb{N}, 1 \leq m\}$ there is an $l \in \mathbb{N}$ s.t. $(g_{n,n+l}(x))_{n+l} = (g_{n,n+l}(y))_{n+l} = 1$. Defining $z \in G_{n+l}$ by putting $(z)_j := (g_{n,n+l}(y))_j$ for $1 \leq j \leq n+l-1$ and $(z)_{n+l} := e$ we can then choose $d_2 := [z]$.

□

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